

Ex: Prove  $n < 2^n$  for all  $n \geq 0$ .

Sol: We proceed by induction on  $n$ .

Base Case: When  $n=0$ , we have  $n=0 < 1 = 2^0 = 2^n$ .

When  $n=1$ , we have  $n=1 < 2 = 2^1 = 2^n$ .

Hence the base cases hold.

~~$n < 2^n$~~   ~~$0 < 2^0$~~   
by themselves not enough

Inductive Step: Suppose  $n < 2^n$  for

some  $n \geq 1$ . Now we compute

$$\underline{n+1} < \underline{2^n} + 1$$

(by IH)

$$\leq 2^n + \underline{n}$$

(by  $n \geq 1$ )

$$< 2^n + 2^n$$

(by IH)

$$= 2(2^n) = 2^{n+1}$$

(by basic arithmetic)

WTS:  $\forall n [P(n) \rightarrow P(n+1)]$

WTS:  $P(n+1)$

i.e.  $n+1 < 2^{n+1}$

Hence  $n+1 < 2^{n+1}$  and we have verified the inductive step.

Hence we conclude the original statement is true by weak mathematical induction.  $\square$

$$A \leftrightarrow B \stackrel{?}{=} (A \rightarrow B) \wedge (\neg A \rightarrow \neg B) \quad \checkmark$$

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (\underline{B \rightarrow A})$$

(Bicond. Exp.)

(Contrapositive)

$$\equiv (A \rightarrow B) \wedge (\neg A \rightarrow \neg B)$$

$$\equiv ((\neg B) \rightarrow (\neg A)) \wedge ((\neg A) \rightarrow (\neg B))$$

(Contrapositive)

$$\equiv (\neg B) \leftrightarrow (\neg A)$$

(Bicond. Exp.)

$$P(0) \wedge \forall k \geq 0 [P(k) \rightarrow P(k+1)] \equiv \forall n \geq 0 [P(n)]$$



We must show

$$\sum_{k=1}^{n+1} k = \frac{1}{2} (n+1)(n+2) \quad \leftarrow \text{Good form}$$

Scrap work

$$\sum_{k=1}^{n+1} k \stackrel{\text{WTS}}{=} \frac{1}{2} (n+1)(n+2)$$

Each is a statement...

$$= \sum_{k=1}^n k + (n+1) \stackrel{\text{WTS}}{=} \frac{1}{2} (n+1)(n+2)$$

BAD Form

$$= \frac{1}{2} n(n+1) + (n+1) \stackrel{\text{WTS}}{=} \frac{1}{2} (n+1)(n+2)$$

$$= (n+1) \left( \frac{1}{2} n + 1 \right) \stackrel{\text{WTS}}{=} \frac{1}{2} (n+1)(n+2)$$

$$= \frac{1}{2} (n+1)(n+2) \stackrel{\text{WTS}}{=} \frac{1}{2} (n+1)(n+2) \quad \checkmark$$

Ex: Prove  $A \cup B = B \cup A$ .  $\leftarrow$   $\star$

Sol 1: Let  $A$  and  $B$  be sets.

("Element Method")

( $\subseteq$ ): Let  $x \in A \cup B$ . Thus either  $x \in A$  or  $x \in B$  by definition of set union. Hence either  $x \in B$  or  $x \in A$  by commutativity of disjunction. Finally,  $x \in B \cup A$  by definition of set union. Hence as  $x$  was

arbitrary,  $A \cup B \subseteq B \cup A$  as desired.

( $\supseteq$ ): (left as an exercise).

Hence  $A \cup B \subseteq B \cup A$  and  $B \cup A \subseteq A \cup B$ , so  $A \cup B = B \cup A$ .  $\square$

Sol 2: Let  $A$  and  $B$  be sets. We have the following:

$$\begin{aligned} \underline{A \cup B} &= \{x: \underline{x \in A \text{ or } x \in B}\} && \text{(by definition of set union)} \\ &= \{x: x \in B \text{ or } x \in A\} && \text{(by commutativity of } \vee \text{)} \\ &= \underline{B \cup A}. && \text{(by definition of union)} \end{aligned}$$

Hence  $A \cup B = B \cup A$  as desired.  $\square$